SINGULAR INTEGRAL OPERATORS ON NON-COMPACT MANIFOLDS AND ANALYSIS ON POLYHEDRAL DOMAINS

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Abstract. We review the definition of a Lie manifold $(M, \mathcal{V})$ and the construction of the algebra $\Psi_{\mathcal{V}}^{\infty}(M)$ of pseudodifferential operators on a Lie manifold $(M, \mathcal{V})$. We give some concrete Fredholmness conditions for pseudodifferential operators in $\Psi_{\mathcal{V}}^{\infty}(M)$ for a large class of Lie manifolds $(M, \mathcal{V})$. These Fredholm conditions have applications to boundary value problems on polyhedral domains and to non-linear PDEs on non-compact manifolds. As an application, we determine the spectrum of the Dirac operator on a manifold with multi-cylindrical ends.

Contents

Introduction

Partial diffferential equations on non-compact manifolds are a common occurrence in Geometry, Group Representations, Mathematical Physics, and other areas of Mathematics and Science. For example, conformally compact manifolds and asymptotically flat manifolds were recently considered in Quantum Gravity and in the study of the AdS-CFT correspondence [6, 7, 15, 25, 36, 60]. One of the main technical issues of the analysis on non-compact manifold M_0 is that an elliptic operator P of order m with elliptic principal symbol is not necessarily Fredholm as an operator $P : H^s(M_0) \to H^{s-m}(M_0)$. In particular, the spectrum of such a P needs not to be discrete.

Analysis on non-compact manifolds plays a role also in the analysis on singular spaces, the non-compact space being the set of regular points endowed with a suitable metric. An important class of singular spaces is provided by polyhedral domains.

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Analysis on polyhedral domains has many features that are not present in the analysis on smooth domains. Several of these issues were discussed for Lipschitz domains in [29, 49, 53, 65]. However, polyhedral domains are not always Lipschitz (recall the "two-brick" example [66]). Moreover, polyhedral domains are amenable to a more detailed analysis [18, 20, 26, 27]. So far, however, this more detailed analysis was devoted mostly to the case of polygonal domains, and, occasionally, to the case of polyhedral domains in space. See however the recent work of Verchota and Vogel on higher dimensional polyhedra [66, 67].

In this paper, we discuss the relevance of singular integral operators in the analysis on non-compact manifolds and in the analysis on polyhedral domains. This paper is largely based on joint results with: Bernd Ammann, Constantin Bacuta, Robert Lauter, Alexandru Ionescu, Marius Mitrea, Bertrand Monthubert, Andras Vasy, Alan Weinstein, Ping Xu, and Ludmil Zikatanov [2, 3, 4, 5, 11, 10, 32, 35, 48, 55]. A central role in the above papers is played by the concept of Lie manifold [3] (their definition is recalled in Definition 2.1) and by the natural pseudodifferential operators acting on a Lie manifold. In [3], the term "manifold with a Lie structure at infinity" was used instead of the term "Lie manifold."

We begin by recalling some results on boundary value problems that motivate our interest in non-compact manifolds. Then we recall the definition of a Lie manifold $(M, \mathcal{V})$, where $\mathcal{V}$ is a suitable Lie algebra of vector fields on M and the construction of the Melrose quantization $\Psi^{\infty}_{\mathcal{V}}(M)$ of the algebra of differential operators naturally associated to a Lie manifold. In addition to reminding some of the necessary results from the above papers, we also prove some new results. We introduce a special class of Lie manifolds ("type I Lie manifolds") in Section 4 and we give some explicit conditions for an operator $P \in \Psi^{\infty}_{\mathcal{V}}(M)$ to be Fredholm if $(M, \mathcal{V})$ is a Lie manifold. Section 5 contains several concrete examples of Lie manifolds and of the use of the Fredholmness conditions for type I Lie manifolds. As a last application, we also determine in Section 6 the essential spectrum of the Dirac operator on a manifold with multi-cylindrical ends. For comparison, let us recall that the essential spectrum of the Laplace operator on a manifold with multi-cylindrical ends is $[0, \infty)$ [35] (solving a conjecture from [45]).

The structure of this paper reflects, to a large extent, the structure of my talk given at the "Conference on Spectal Geometry of Manifolds with Boundary and Decomposition of Manifolds," organized by B. Booss-Bavnbek, G. Grubb, and K. Wojciechowski, whom I thank for their efforts and for the opportunity to present my results. This paper, however, contains more precise statements and several new results. I also thank Bernd Ammann, Constantin Băcuţa, Craig Evans, Alexandru Ionescu, Robert Lauter, and Irina Mitrea for useful discussions. We shall write ":=" for "the left hand side is equal by definition to the right hand side."

1. Boundary value problems

Let $\Omega \subset \mathbb{R}^n$ be a *bounded, open* set with boundary $\partial\Omega := \overline{\Omega} \smallsetminus \Omega$.. Let us consider on Ω "simplest" boundary value problem, the Poisson problem

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = g. \end{cases} \tag{1}$$

A well known, classical results [22, 64] is the following "shift theorem" (or "regularity theorem").

Theorem 1.1 (Classical). *If $\partial\Omega$ is smooth, then $\tilde{\Delta}(u) = (\Delta u, u|_{\partial\Omega})$ defines an isomorphism*
$$\tilde{\Delta} : H^{m+2}(\Omega) \to H^m(\Omega) \oplus H^{m+3/2}(\partial\Omega),$$
for $m \in \mathbb{R}$, $m \geq -1$.

Although this is not needed in this paper, let us notice, for the interested reader, that the range of m in the above theorem can be improved to $m \in \mathbb{R}$. See for example [9] and the references therein. This improvement is relevant in some problems arising in the applications of elasticity to Engineering [63].

It follows right away from the above theorem that if f, g, and $\partial\Omega$ are smooth, then u is also smooth (including the boundary). This is, however, not true in general if $\partial\Omega$ is not smooth. In particular, the above theorem is not true if $\partial\Omega$ is not smooth. Indeed, let us take Ω to be the square $(0,1)^2$ and $g = 0$ and let us assume that u is smooth. Then $\partial_x^2 u(0,0) = 0 = \partial_y^2 u(0,0)$, and hence $f(0,0) = \Delta u(0,0) = 0$. Thus no solution of our problem (1) on the unit square $\Omega = (0,1)^2$ is smooth if $g = 0$ and $f(0,0) \neq 0$. The same problem arises on any polygonal domain. A detailed and far reaching analysis of the above issues can be found in the fundamental paper of Jerison and Kenig [29], which shows the exact range of applicability of the above theorem on a Lipschitz domain in the plane.

From a practical point of view, the fact that the above "shift theorem" does not extend directly to non-smooth domains is quite inconvenient for applications. More precisely, if one want to solve an elliptic partial differential equation using the finite element method and a quasi-uniform mesh, the rate of convergence of the method is governed by the smoothness of the solution. In particular, one achieves only low orders of convergence using quasi-uniform meshes on a polygon [68]. The problem, however, is due to the use of quasi-uniform meshes (and the use of the usual, isotropic Sobolev spaces). Indeed, it was shown by Babuška already in the '70 [8] that one can achieve the same rate of convergence as for smooth domains, provided that one chooses correctly the finite element space. See also [11] and [57].

In this paper we look at these issues from the point of view of Lie manifolds. (We shall recall the definition of Lie manifolds and their relevance to boundary value problems below.) Let us begin by discussing the relatively simpler example of a polygonal domain (or, more generally, of a domain whose boundary has conical points).

One of the most successful approaches so far is to use polar coordinates (r, θ) around the vertices of a polygon. For a general domain with conical points, one uses generalized polar coordinates. In the mathematical community this approach was pioneered by Kondratiev [30], but in the Engineering community the use of polar coordinates and of the Mellin transform is apparently much older.

To explain this approach, let us consider the open angle $\Omega = \{\theta \in (0, \alpha)\}$ and polar coordinates, then
$$\Delta = r^{-2}\big((r\partial_r)^2 + \partial_\theta^2\big).$$
This suggests to look at differential operators on Ω of the form
$$\sum_{i+j \leq m} a_{ij}(r,\theta)(r\partial_r)^i \partial_\theta^j \tag{2}$$
with a_{ij} smooth. Operators of this type are called *totally characteristic operators*, and they are defined on any manifold with boundary (see below). The relevant part of the boundary in our example is given by $r = 0$.

Analogously, we define then the *totally characteristic vector fields* on $[0,\infty) \times (0,\alpha) \ni (r,\theta)$ to be the vector fields X of the form

$$X = a(r,\theta) r\partial_r + b(r,\theta)\partial_\theta, \tag{3}$$

where a and b are smooth functions. An important observation [44, 45], is that the totally characteristic vector fields form a Lie algebra. This observation extends to polygonal domains, domains with conical points, and, more generally, to polyhedral domains.

Before explaining our use of vector fields, let us take a look at two more examples. Let us consider the "edge" $\Omega \times \mathbb{R}$, where $\Omega = \{\theta \in (0,\alpha)\}$, as above. If we consider cylindrical coordinates (r,θ,z) in $\mathbb{R}^3$, then the Laplace operator becomes

$$\Delta = r^{-2}\big((r\partial_r)^2 + \partial_\theta^2 + r^2\partial_z^2\big).$$

If we ignore the coefficient r^{-2}, we are lead to consider differential operators generated by products of the derivatives $r\partial_r$, ∂_θ, and $r\partial_z$ (and smooth coefficients). The differential operators of this kind that are vector fields are of the form

$$X = a(r,\theta,z) r\partial_r + b(r,\theta,z)\partial_\theta + c(r,\theta,z) r\partial_z, \tag{4}$$

with a, b, and c smooth functions. These vector fields ("edge–type vector fields") form also a Lie algebra.

2. Lie algebras of vector fields

The examples of the previous section, among others, have led Melrose to formulate a program to study the analysis of differential operators generated by suitable Lie algebras of vector fields [44, 45]. Many important results in this program were obtained by [21, 31, 33, 41, 45, 46, 61, 62, 69]. In this paper, however, we shall be mainly concerned with the approach to this program developed in [3, 4, 32, 35, 50, 55].

We shall consider a compact manifold with corners M together with a subspace $\mathcal{V} \subset \Gamma(TM)$, consisting of vector fields tangent to all faces of M and satisfying certain conditions that make $(M,\mathcal{V})$ a "Lie manifold." We shall denote by M_0 the interior of M and by ∂M the set of boundary points of M. In particular, $M_0 = M \smallsetminus \partial M$. The following definition is essentially from [45], but it was formalized in [3].

Definition 2.1. Let M be a manifold with corners. A *Lie manifold* is a pair $(M,\mathcal{V})$, where $\mathcal{V}$ is a set of vector fields tangent to all faces of M satisfying the following conditions:

(i) $\mathcal{V}$ is closed under the Lie bracket $[\ ,\ ]$;
(ii) $\mathcal{C}^\infty(M)\mathcal{V} = \mathcal{V}$;
(iii) $\mathcal{V}$ is linearly generated locally in the neighborhood of each point $p \in M$ by n linearly independent vector fields $X_1,\ldots,X_n$ with $\mathcal{C}^\infty(M)$ coefficients.
(iv) If in the conditions above $p \in M_0$, then the vector fields $X_1,\ldots,X_n$, locally generating $\mathcal{V}$ around p, also give a local basis of TM around p.

Condition (iii) means the following. For each $p \in M$ there exists an open neighborhood U of p in M and vector fields $X_1,\ldots,X_n \in \mathcal{V}$ such that for any $X \in \mathcal{V}$, there exist uniquely determined smooth functions $a_1,\ldots,a_n$ such that

$$X = \sum a_i X_i \text{ on } U. \tag{5}$$

The following remark is slightly less elementary, but it will be useful in several places. It also explains the above definition.

Remark 2.2. It follows from the last axiom that the integer n appearing above must be the same as the dimension of M. In particular, $\mathcal{V}$ is a $\mathcal{C}^\infty(M)$–module isomorphic to a direct summand of the free $\mathcal{C}^\infty(M)$–module $\mathcal{C}^\infty(M)^N$, for some N. That is $\mathcal{V}$, is a projective $\mathcal{C}^\infty(M)$–module. Then the Serre-Swan theorem states that there exists a vector bundle $A \to M$, unique up to isomorphism, such that $\mathcal{V}$ is isomorphic, as a $\mathcal{C}^\infty(M)$-module to $\Gamma(A)$, the space of sections of A. The definition of $\mathcal{V}$ as a space of vector fields on M and the naturality of A, show that there exists a vector bundle map $\varrho : A \to TM$, called *anchor map*, that endows A with the structure of a Lie algebroid. We shall therefore call A the *Lie algebroid associated to the Lie manifold* $(M, \mathcal{V})$. Condition (iv) of Definition 2.1 is then equivalent to saying that ϱ is an isomorphism on the interior of M. See [3, 35] for more details.

In [3], the manifolds introduced in the above definition were called "manifolds with a Lie structure at infinity."

Define $\mathrm{Diff}_{\mathcal{V}}(M)$ to be the algebra of differential operators on M generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$. The differential operators in $\mathrm{Diff}_{\mathcal{V}}(M)$ are the singular differential operators we plan to study in this paper, due to their applications to analysis on singular domains and on non-compact manifolds.

Even if one is primarily interested in differential operators, in order to invert them, one has to consider also integral kernel operators. In our case, these integral kernel operators will be pseudodifferential operators. To see their relevance for boundary value problems, in particular, let us quickly recall the method of layer potentials.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain (for example, a domain with piecewise C^1-boundary). Let $c_n^{-1} = \omega_n(2-n)$, where ω_n is the surface of the unit sphere in $\mathbb{R}^n$), $\nu(y)$ is the outer unit normal, and $d\sigma(y)$ is the induced measure on $\partial\Omega$. Then the operator

$$Kf(x) = c_n \int_{\partial\Omega} \frac{(y-x)\cdot\nu(y)}{|y-x|^n} f(y) d\sigma(y),$$

can be used to determine the boundary value of the double layer potential operator (the double layer potential operator is the extension of the formula for K to x in the interior of Ω, while $y \in \partial\Omega$). If one can establish the invertibility of $\frac{1}{2}I + K$ as a pseudodifferential operator on $\partial\Omega$, then one obtains that the boundary value problem (1) has a solution for $f \in H^{1/2}(\partial\Omega)$, $g = 0$, which can be then used to obtain a solution for more general data.

If $\partial\Omega$ is smooth, then K is a pseudodifferential operator of order -1. Hence $\frac{1}{2}I + K$ is a Fredholm operator of index zero, because $\partial\Omega$ is also compact. Therefore $\frac{1}{2}I + K$ is invertible if, and only if, it is injective or surjective. The injectivity can usually be checked using energy methods. This completes our very brief summary of the method of layer potentials for smooth domains.

The above reasoning does not extend directly to the case when $\partial\Omega$ is not smooth, because K may fail to be compact [23, 24]. See also [65]. Nevertheless, for the case of a polygon, K is in a class of operators that is well understood (the class of Hardy-type operators), see [24, 37]. An approach to the study of Hardy-type operators is provided by the operators in the "b-calculus" on $\partial\Omega$ [37, 45] to which the Hardy-type operators are closely related. The b-calculus is the pseudodifferential analog

of totally-characteristic differential operators. Using an iterative argument, one can show that the method of layer potentials extends domains with conical points (hence to to curvilinear polygons as well) [23, 24, 37, 48]. Let r denote the distance the set of singularities on the boundary (*i.e.,* the distance to the vertices, if our domain is a polygon, or the distance to the conical points, if our domain is a domain with conical points). Then define

$$r^a H_b^m(\Omega) := \{u \in L^2_{loc}(\Omega), r^{-a-1+|\alpha|}\partial^\alpha u \in L^2(\Omega), |\alpha| \leq m\}.$$

Theorem 2.3. *Let Ω be a polygon or a domain with conical points. Then there exists $\eta > 0$ such that the map $\tilde{\Delta}(u) = (\Delta u, u|_{\partial\Omega})$ establishes an isomorphism*

$$\tilde{\Delta} : r^a H_b^{m+2}(\Omega) \to r^{a-2} H_b^m(\Omega) \oplus r^{a-2} H_b^{m+3/2}(\partial\Omega),$$

for all $|a| < \eta$.

See [11, 30, 48] or [52]. If Ω is a polygon with maximum angle α_M, then we can choose $\eta = \pi/\alpha_M$.

For a convex polytope Ω, K will be an integral operator in a distinguished class of pseudodifferential operators on the boundary $\partial\Omega$, a class closely related to $\mathrm{Diff}_{\mathcal{V}}(\partial\Omega)$, for a suitable Lie algebra of vector fields $\mathcal{V}$ on $\partial\Omega$. In Melrose's terminology, this class of pseudodifferential operators "quantizes" $\mathrm{Diff}_{\mathcal{V}}(\partial\Omega)$. These operators can be thought of as "singular pseudodifferential operators on $\partial\Omega$." One is lead therefore to consider the following problem, which we have dubbed "Melrose's quantization problem," [44]:

> **Melrose's quantization problem:** *Given a Lie manifold $(M, \mathcal{V})$, one wants to construct $\Psi^\infty_{\mathcal{V}}(M)$, an algebra of pseudodifferential operators on M with the symbolic and analytic properties similar to those of the algebra of pseudodifferential operators on a compact manifolds and such that all differential operators in $\Psi^\infty_{\mathcal{V}}(M)$ be generated by $\mathcal{V}$.*

If $M = \partial\Omega$, then a variant of the algebra $\Psi^\infty_{\mathcal{V}}(M)$ should contain the operator K and be compatible with (i.e. quantize) $\mathrm{Diff}_{\mathcal{V}}(\partial\Omega)$, thus generalizing the Hardy type operators and the b-calculus. Below, we shall give a construction of the Lie manifold $(M, \mathcal{V})$ associated to a convex polytope Ω.

3. Melrose's quantization problem

We propose a geometric solution in Melrose's spirit. This solution, given in [4], requires the choice of an appropriate metric on $M_0 = M \smallsetminus \partial M$, the interior of M. More precisely, we choose on $M_0 = M \smallsetminus \partial M$ a metric g_0 that has in a neighborhood of any point $x \in M$, a local orthonormal basis given by sections of $\mathcal{V}$. A metric g_0 on M_0 with this property will be called *compatible* (with the Lie manifold structure $(M, \mathcal{V})$). For points $x \in M_0$, the above condition defining a compatible metric is automatically satisfied, as it follows from Condition (iv) of Definition 2.1.

The definition of a compatible metric on M_0 can be reformulated as follows. Let $A \to M$ be the Lie algebroid of $(M, \mathcal{V})$, that is, the vector bundle such that $\Gamma(A) \simeq \mathcal{V}$ as $\mathcal{C}^\infty(M)$-modules. See our discussion after Definition 2.1. Then any metric on A defines, by restriction, a metric on TM_0. The resulting metric g_0 on M_0 is a compatible metric, and any compatible metric arises in this way. The metric g_0 is *not* the restriction of a smooth metric on M, in fact, g_0 will be singular on M.

Let

$$(x, y) \mapsto (x, \tau(x, y)) \in TM_0$$

be a local inverse of the Riemannian exponential map $TM_0 \ni v \mapsto \exp_x(-v) \in M_0 \times M_0$. Let

$$\big[a_\chi(D)u\big](x) = (2\pi)^{-n} \int_{M_0} \left(\int_{T_x^* M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x, y)) a(x, \eta) u(y)\, d\eta \right) dy.$$

Let $S^m(A^*)$ denote the space of symbols of type $(1, 0)$ (*i.e.,* satisfying Hörmander's usual estimates [28]).

Definition 3.1. We define $\Psi_{\mathcal{V}}^\infty(M)$ to be the space of pseudodifferential operators $\mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(M_0)$ linearly generated by $a_\chi(D)$ and $b_\chi(D)\exp(X_1)\dots\exp(X_k)$, where $a \in S^\infty(A^*)$, $b \in S^{-\infty}(A^*)$, and $X_j \in \mathcal{V}$.

The above definition is consistent with the general principle that all quantities on M_0 (functions, Sobolev spaces) should be defined using the metric g_0. See also [45]. This is, in fact, what leads to the definition of the spaces $H_b^m(\Omega)$ and $H_b^m(\partial\Omega)$ before Theorem 2.3.

Melrose's quantization problem has a solution [4] (see also [35, 50, 55]. Important related results were obtained by [21, 31, 33, 41, 45, 46, 61, 62, 69]. Let $\mathrm{Diff}(M_0)$ denote all differential operators on M_0.

Theorem 3.2 (Ammann-Lauter-Nistor)**.** *The space $\Psi_{\mathcal{V}}^\infty(M)$ is an algebra of pseudodifferential operators that "quantizes" the Lie algebra $\mathcal{V}$, in the sense that $\Psi_{\mathcal{V}}^\infty(M)$ has the usual symbolic and analytic properties that pseudodifferential operators have on compact manifolds, and $\Psi_{\mathcal{V}}^\infty(M) \cap \mathrm{Diff}(M_0) = \mathrm{Diff}_{\mathcal{V}}(M)$. In particular, there exist surjective principal symbol maps $\sigma^{(m)} : \Psi_{\mathcal{V}}^m(M) \to S^m(A^*)$ with kernel $\Psi_{\mathcal{V}}^{m-1}(M)$ and any $P \in \Psi_{\mathcal{V}}^m(M)$ defines a continuous map $H^s(M_0) \to H^{s-m}(M_0)$.*

By slightly enlarging the construction of the algebra $\Psi_{\mathcal{V}}^m(M)$ by including some additional regularizing operators, we recover the Hardy type operators as well as the (small) b-calculus.

The most difficult part in the proof of the above theorem is to show that $\Psi_{\mathcal{V}}^\infty(M)$ is closed under composition. Our proof in [4] is to show that $\Psi_{\mathcal{V}}^\infty(M)$ is the homomorphic image of $\Psi^\infty(\mathcal{G})$. Here $\mathcal{G}$ is a groupoid integrating the Lie algebroid A associated to M [19, 54] and $\Psi^\infty(\mathcal{G})$ is algebra of pseudodifferential operators [55] (in particular, it is closed under composition). See also [16, 50]. The groupoid $\mathcal{G}$ plays the role of a kernel space, because $\Psi^\infty(\mathcal{G}) = I_c^\infty(\mathcal{G}, M)$, the space of compactly supported distributions on $\mathcal{G}$ that are conormal to M. These kernel spaces are very closely related to a construction of Melrose (the stretched b-product ${}^bM^2$), see [44, 45]. It is, in general, a difficult task to find a groupoid integrating a Lie algebroid A, and, in fact, this is not always possible. It is a deep theorem of Crainic and Fernandez that this is possible for the Lie algebroids associated to Lie manifolds [19]. See also [54], which suffices for example for the examples considered in next section.

4. An application to Fredholm conditions

We shall now obtain some criteria for operators $P \in \Psi_{\mathcal{V}}^m(M)$ to be Fredholm. This has applications to boundary value problems as well as to non-linear partial differential equations on non-compact manifolds.

We define the Sobolev space $H^s(M_0)$ to be the domain of $(1+d^*d)^{s/2}$, where d is the de Rham differential and $s \geq 0$. For $s < 0$, we use duality to define $H^s(M_0)$. Also, an elliptic operator $P \in \Psi^m_{\mathcal{V}}(M)$ is one for which $\sigma^{(m)}(P)(\xi) \neq 0$ for all $\xi \in A^*$, $\xi \neq 0$.

The main theorem, which will ocupy us the rest of this section, is the following ("type I Lie manifolds," as well as the rest of the unexplained notation of the following theorem, are introduced below).

Theorem 4.1. *Let $(M, \mathcal{V})$ be a type I Lie manifold. Assume each hyperface of M has a defining function. If $P \in \Psi^m_{\mathcal{V}}(M)$, then there exist pseudodifferential operators P_α on $M_\alpha \times G_\alpha$, invariant with respect to right translations by G_α such that $P : H^s(M_0) \to H^{s-m}(M_0)$ is Fredholm if, and only if, P is elliptic and all $P_\alpha : H^s(M_\alpha \times G_\alpha) \to H^{s-m}(M_\alpha \times G_\alpha)$ are invertible for all $\alpha \neq 0$.*

This theorem will follow from the results of [32, 35]. More general Fredholmness conditions were obtained in [32, 35], but they involve some conditions that may be difficult to use. On the other hand, the conditions for the above theorem are easier to check.

We shall assume that our Lie manifold $(M, \mathcal{V})$ satisfies the following four conditions. Our first condition is that there exist, for any (closed) face $F \subset M$, a fibration $p_F : F \to B_F$ with connected fibers, such that

$$\varrho(A_p) = T_p p_F^{-1}\big(p_F(p)\big), \text{ for any } p \in F_0 := \overset{\circ}{F}. \tag{6}$$

We shall use the Lie algebroid $A \to M$ associated to $\mathcal{V}$ and the anchor map $\varrho : A \to TM$ introduced in Remark 2.2.

Another way of formulating our first condition, Equation (6), is that, for any p in the interior of F, the tangent space at p through the fiber of p_F containing p coincides with the set $X(p)$, $X \in \mathcal{V}$. Yet another way of formulating this condition is that the set $\{\exp_X(p)\}$ is the fiber of p_F containing p, where $\exp_{tX}$ is the one-parameter group of diffeomorphisms obtained by integrating X and p is in F_0, the interior of F. (See [3] for the easy proof that $\exp_{tX}$ is defined for all t.) From this condition it also follows that the isotropy Lie algebras

$$\mathfrak{l}_p := \ker(\varrho : A_p \to T_p M), \quad p \in F_0, \tag{7}$$

have the same dimension, and hence they define a vector bundle on the interior of F. Let $\mathfrak{L}_F \to F_0$ denote this vector bundle. Then $\mathfrak{L}_F$ is a bundle of Lie algebras.

For any $p \in M$, the vector space $\mathfrak{l}_p$ (see Equation (7)) has a natural structure of Lie algebra. (To see this, let $X, Y \in \mathcal{V}$ be such that $X(p) = Y(p) = 0 \in T_p M$. Then $[X, Y](p)$ depends only on $X(p)$ and $Y(p)$. See also [39].) We shall call $\mathfrak{l}_p$ *the isotropy Lie algebra of p.* Let G_p be the simply connected Lie group with Lie algebra $\mathfrak{l}_p$. We say that G_p is an exponential Lie group if the exponential map defines a diffeomorphism

$$\exp : \mathfrak{l}_p \simeq G_p. \tag{8}$$

The simply-connected nilpotent Lie groups and most simply-connected solvable Lie groups satisfy this condition. Our third condition is then

$$G_p \textit{ is a solvable, exponential group.} \tag{9}$$

To formulate the third condition, let M_α be the orbits of the diffeormorphisms $\exp_X$, $X \in \mathcal{V}$, acting on M, where α belongs to an index set $\mathfrak{I}$ containing 0. By

our first assumption, if $p \in M_\alpha$ is an interior point of a face F, then M_α coincides with the interior of the fiber of $p_F : F \to B_F$ containing p. In particular, $M_\alpha = M_0$ if $\alpha = 0$. Our third assumption is that there exists a bundle of Lie algebras $\mathfrak{A}_F$ on the interior of B_F such that

$$\mathfrak{L}_F \simeq p_F^* \mathfrak{A}_F. \tag{10}$$

Let $\mathfrak{A}_{Fq}$ be the fiber of $\mathfrak{A}_F$ above q, with q in the interior of B_F. Let G_q be a simply-connected Lie group with Lie algebra $\mathfrak{A}_{Fq}$. Let $q = p_F(p)$. Since $G_q \simeq G_p$ and $\mathfrak{A}_{Fq} \simeq \mathfrak{l}_p$, it follows from our second assumption (Equation (9)) that the exponential map

$$\exp : \mathfrak{L}_{Fq} \to G_q \tag{11}$$

is a diffeomorphism.

We shall also need differentiable groupoids. Let say first that a groupoid is a "group with several units," and that the product of two elements is defined only if the domain of the first matches the range of the second one. The model for a groupoid is a set of bijective functions. More precisely, a groupoid is a small category all of whose elements are invertible. (A category is small if the class of its objects is in fact a set.) See [35, 55] for an introduction to differentiable groupoids that is suitable for our purposes.

Let $p_F : G_F \to \overset{\circ}{B}_F$ be $\mathfrak{A}_F$ as a manifold, but with the Lie group structure on each fiber induced by the exponential map (which is a diffeomorphism, see Equation (11)). For each α, let $q = p_F(M_\alpha)$. We shall denote by G_α the fiber of G_F above q. (This makes sense in view of our first assumption, Equation (6).) Consider the fibered product

$$\begin{aligned} \mathcal{G}_F &= F_0 \times_{B_F} F_0 \times_{B_F} G_F \\ &:= \{(x,y,g) \in F_0 \times F_0 \times G_F, p_F(x) = p_F(y) = p_F(g) \in B_F\}, \end{aligned} \tag{12}$$

with the groupoid structure given by the product $(x, y, g)(y, z, h) = (x, z, gh)$, the set of units F_0, and the domain map $d(x, y, g) = y \in F_0$ and range map $r(x, y, g) = y \in F_0$. Then $\mathcal{G}_F$ is a differentiable groupoid. Our fourth assumption is that

$$\mathcal{G} := \cup \mathcal{G}_F \textit{ is a Hausdorff differentiable groupoid with Lie algebroid } \simeq A. \tag{13}$$

The groupoid $\mathcal{G}$ is the disjoint union of the groupoids $\mathcal{G}_F$, and the structural morphisms (composition, domain, range, ...) are the ones induced from $\mathcal{G}_F$. In particular, the set of units of $\mathcal{G}$ is the disjoint union of the units of the groupoids $\mathcal{G}_F$, that is, $\mathcal{G}$ has as a set of units $M = \cup F_0$. By the results of [54], there is at most one differentiable structure on $\mathcal{G}$ with Lie algebroid A. It induces the given differentiable structure on $\mathcal{G}_F$.

Definition 4.2. A Lie manifold satisfying the above four conditions (Equations (6), (9), (10), and (13)) will be called *a type I Lie manifold.*

For the proof of Theorem 4.1 it is necessary to recall a few constructions and to prove some intermediate results.

Recall [35, 55] that $P \in \Psi^\infty(\mathcal{G})$ is in fact a family of pseudodifferential operators $P = (P_x)$, $x \in M$, with P_x acting on

$$\mathcal{G}_x := d^{-1}(x) = \{(z, x, g), z \in M_\alpha,\ e \text{ the unit of } G_\alpha\} \simeq M_\alpha \times G_\alpha$$

and satisfying some additional assumptions. These additional conditions are: right invariance for multiplication by elements in $\mathcal{G}$, that the family (P_x) be a smooth family, and a suport condition that implies, in particular, that all P_x are properly supported.

If $x, y \in M_\alpha$, then right multiplication by (x, y, e) is a diffeomorphism $\mathcal{G}_x \to \mathcal{G}_y$ mapping P_x to P_y, by the assumption of right invariance. The restriction $\tilde{P}_\alpha$ of P to $\mathcal{G}_x$ is simply P_x, where $x \in M_\alpha$. The canonical isomorphism $\mathcal{G}_x \to M_\alpha \times G_\alpha$ will map

$$P_x \to P_\alpha \tag{14}$$

for all operators P_x, with P_α a pseudodifferential operator on $M_\alpha \times G_\alpha$ that is independent of x. This operator is right invariant with respect to the action of G_α, again by the invariance condition.

Let $\phi \in \mathcal{C}_c^\infty(\mathcal{G})$. We shall denote by ϕ_x the restriction of ϕ to $\mathcal{G}_x$. We fix a metric on A which will fix a metric on each of the spaces $\mathcal{G}_x$ and hence a volume form smoothly depending on x. We shall denote by $\| \cdot \|$ the norm on $L^2(\mathcal{G}_x)$ or the norm of a bounded opearator on this space, for any x. There will be no danger of confusion. We begin by examining the consequences of the assumption that $\mathcal{G}$ is Hausdorf.

Proposition 4.3. *Let $P = (P_x) \in \Psi^m(\mathcal{G})$. Then $\|P_x \phi_x\|$ depends continuously on $x \in M$, for any $\phi \in \mathcal{C}_c^\infty(\mathcal{G})$.*

Proof. Fix $\phi \in \mathcal{C}_c^\infty(\mathcal{G})$. From the definition of the algebra $\Psi^\infty(\mathcal{G})$, it follows that there exists a function $\psi \in \mathcal{C}_c^\infty(\mathcal{G})$ such that $\psi_x = P_x \phi_x$. The continuity of the function $\|\psi_x\|$ follows from the assumption that $\mathcal{G}$ is Hausdorf and from the smoooth dependence of the measure on $\mathcal{G}_x$ on x. □

We now prove as a consequence the following result.

Corollary 4.4. *Let $P = (P_x) \in \Psi^m(\mathcal{G})$ and $x \in M_0$. If $P_x = 0$, then $P_y = 0$, for any $y \in M$. That is $P = 0$.*

Proof. This is a consequence of the fourth Assumption, namely Equation (13). Indeed, assume $P_x = 0$ for some $x \in M_0$. Then $P_y = 0$ for all $y \in M_0$, by the right invariance of the operators P_x. To prove that $P_y = 0$ for some arbitrary y, we now show that $P_y \eta = 0$ for any $\eta \in \mathcal{C}_c^\infty(\mathcal{G}_y)$. Let $\phi \in \mathcal{C}_c^\infty(\mathcal{G})$ that restricts to η on $\mathcal{G}_y$ (*i.e.,* $\phi_y = \eta$). This is possible since $\mathcal{G}_y$ is a closed subset of the Hausdorf, locally compact space $\mathcal{G}$. Then $\|P_y \phi_y\|$ is a continuous function of $y \in M$ that vanishes for $y \in M_0$. Since M_0 is dense in M, we obtain that $P_y \phi_y = 0$ for all y. □

The assumption that $\mathcal{G}$ is Hausdorff therefore implies that the natural action of $\Psi^\infty(\mathcal{G})$ on $\mathcal{C}_c^\infty(M_0)$ is faithful (*i.e.,* the induced morphism $\Psi^\infty(\mathcal{G}) \to \mathrm{End}(\mathcal{C}_c^\infty(M_0))$ is injective). Fix $z \in M_0$ and consider the canonical bijection (diffeomorphism) $\mathcal{G}_z \to M_0$. Then the map

$$\Psi^\infty(\mathcal{G}) \ni P = (P_x) \to P_z \in \Psi^\infty(\mathcal{G}_z) \simeq \Psi^\infty_{\mathcal{V}}(M) \tag{15}$$

is a bijection. We shall henceforth identify these two algebras (this is incidentaly the canonical surjection constructed in [4]). In particular, we can define P_α, for any $P \in \Psi^\infty_{\mathcal{V}}(M) = \Psi^\infty(\mathcal{G})$ using Equation (14).

Corollary 4.5. *We have that $(PQ)_\alpha = P_\alpha Q_\alpha$, for all $P, Q \in \Psi^\infty_{\mathcal{V}}(M)$.*

Proof. The product in the algebra $\Psi^\infty(\mathcal{G})$ is $PQ = (P_x Q_x)$, if $P = (P_x)$ and $Q = (Q_x)$, $x \in M$. The result then follows from the definition of P_α given in Equation (14). $\square$

Yet another corollary of Proposition 4.3 is the following.

Corollary 4.6. *Let $P = (P_x) \in \Psi^0(\mathcal{G})$. Then the function $M \ni x \to \|P_x\| \in \mathbb{R}$ is lower semi-continuous.*

Proof. Indeed, let $\alpha \in \mathbb{R}$. We need to show that the set $\{x \in M, \|P_x\| > \alpha\}$ is open in M. Let $y \in M$ be such that $\|P_y\| > \alpha$. Then we can find $\eta \in \mathcal{C}_c^\infty(\mathcal{G}_y)$ such that $\|P_y \eta\| > \alpha \|\eta\|$. Let $\phi \in \mathcal{C}_c^\infty(\mathcal{G})$ be such that $\phi_y = \eta$. Since $\|P_x \phi_x\|$ and $\|\phi_x\|$ are continuous, we have that $\|P_x \phi_x\|/\|\phi_x\|$ is well defined and continuous in a neighborhood of y. But then $\|P_x \phi_x\|/\|\phi_x\| > \alpha$ defines an open neighborhood of y on which $\|P_x\| > \alpha$. $\square$

This in turn gives the following.

Corollary 4.7. *For any $P_x \in \Psi^0(\mathcal{G})$ we have $\|P_y\| \leq \|P_x\|$ for any $x \in M_0$, $y \in M$. In other words, the function $M \ni y \to \|P_y\| \in [0, \infty)$ attains its maximum at any point $x \in M_0$.*

Proof. All operators P_x are unitarily equivalent for $x \in M_0$. Therefore the function $M \ni y \to \|P_y\| \in [0, \infty)$ is constant on M_0. Now if $\|P_y\| > \|P_x\|$ for some $y \in \partial M = M \smallsetminus M_0$ and some $x \in M_0$, then, by choosing $\|P_y\| > \alpha > \|P_x\|$, we contradict the fact that the set $\{y \in M, \|P_y\| > \alpha\}$ is open in M. $\square$

Let $\overline{\Psi}_{-\infty}$ be the closure of the ideal $\Psi_{\mathcal{V}}^{-\infty}(M)$ in the family of norms of operators $H^t(\mathcal{G}_x) \to H^r(\mathcal{G}_x)$, $x \in M$. By Corollary 4.7, this closure is the same as the closure of $\overline{\Psi}_{-\infty}$ in the topology of continuous operators $H^t(M_0) \to H^r(M_0)$. Let $\Psi^s := \Psi_{\mathcal{V}}^s(M) + \Psi^{-\infty}$. Then $\overline{\Psi}_s \overline{\Psi}_{s'} \subset \overline{\Psi}_{s+s'}$.

Denote by $\mathcal{L}(\mathcal{H})$ the set of continuous linear operators $\mathcal{H} \to \mathcal{H}$. Let us notice that the exact sequence of envelopping C^*-algebras of groupoids (see for example [35][Equation 16]) or the structure theorem [35][Theorem 4.4] show that the natural representation $C^*(\mathcal{G}) \to \mathcal{L}(\mathcal{H})$ is injective. In particular, $\mathcal{G}$ is amenable. Then Theorems 4.1 and 4.8 follow right away from [35][Theorem 9.]. We prefer however to include some arguments, to make the paper more complete. Let $\mathfrak{A}(M)$ be the norm closure of $\Psi_{\mathcal{V}}^0(M)$ acting on $L^2(M_0)$. The Theorems 4.1 and 4.8 remain true for $P \in \mathfrak{A}(M)$ and $m = 0$.

We shall need also the following theorem.

Theorem 4.8. *We keep the assumptions and notation of Theorem 4.1 and fix $s \in \mathbb{R}$. Let $P \in \Psi_{\mathcal{V}}^m(M)$, then $P : H^s(M_0) \to H^{s-m}(M_0)$ is compact if, and only if, $\sigma^{(m)}(P) = 0$ and $P_\alpha = 0$, for all $\alpha \neq 0$.*

Proof. This will be a consequence of the results of [32, 35]. As in [5][Proposition 5.2 and Theorem 6.2], we can find an invertible pseudodifferential operator $P \in \overline{\Psi}_r$, for any r. ("Invertible" here means that the inverse is in $\overline{\Psi}_{-r}$.) This allows us to assume that P has order zero.

Let x_H be a defining function for each hyperface H of M and x the product of all defining functions of hyperfaces of M. Let $P \in \Psi_{\mathcal{V}}^{-1}(M)$. If $P_\alpha = 0$ for all $\alpha \neq 0$, then $P = xQ$, with $Q \in \Psi_{\mathcal{V}}^{-1}(M)$. Therefore P maps $H^s(M_0)$ continuously to $xH^{s-1}(M_0)$. Since $xH^{s-1}(M_0) \to H^s(M_0)$ is a compact map (see, for example,

[2][Theorem 3.6] for this easy generalization of Kondrachov's theorem), it follows that $P : H^s(M_0) \to H^s(M_0)$ is compact.

As above, we can assume that P has order zero. Suppose now that $P : H^s(M_0) \to H^s(M_0)$ is compact. Then $\sigma^{(0)}(P) = 0$, as in the classical case [28]. Assume, by contradiction, that $P_\alpha \neq 0$, for some α. Fix for the rest of this discussion $x \in M_\alpha$ and $\phi \in \mathcal{C}_c^\infty(\mathcal{G}_x)$, $\mathcal{G}_x = M_\alpha \times G_\alpha$, such that $P\phi \neq 0$. We extend ϕ to a smooth, compactly supported function on $\mathcal{G}$, still denoted by ϕ. Let ϕ_y, $y \in M_0$ be the restriction of ϕ to $\mathcal{G}_y \simeq M_0$. As $y \to x$, $y \in M_0$, we have that $\phi_y \to 0$ weakly, but $\|P\phi_y\| \to \|P_x\phi_x\| \neq 0$. So P cannot be compact. □

We shall need also the following corollary of the above proof.

Corollary 4.9. *We keep the notation of the proof of Theorem 4.8.*
(i) $(PQ)_\alpha = P_\alpha Q_\alpha$ *for* $P, Q \in \overline{\Psi}_\infty$.
(ii) *Assume that* $P \in \Psi_{\mathcal{V}}^m(M)$ *is a Fredholm operator* $P : H^s(M_0) \to H^{s-m}(M_0)$. *Then there exist* $Q \in \overline{\Psi}_{-m}$ *such that* $PQ - I$ *and* $QP - I$ *are compact operators.*

Proof. Part (i) is clear by the definition of $\overline{\Psi}_\infty$ and Corollary 4.7.

As in the above proof, we can assume that P has order zero. It was proved in [5] and in [32] that $\overline{\Psi}_0$ is closed under holomorphic functional calculus. Since we can construct Q out of P using holomorphic functional calculus, it follows that $Q \in \overline{\Psi}_0$. □

We are ready now to prove the main result of this section, Theorem 4.1.

Proof. Assume that $\sigma^{(0)}(P)(\xi)$, $\xi \neq 0$, and P_α, $\alpha \neq 0$, are invertible. The structure theorems of [32, 35] show that the map $\overline{\sigma}$

$$\Psi_{\mathcal{V}}^0(M) \ni P \to \big(\sigma^{(0)}(P)|_{S^*A}, P_\alpha\big) \in C(S^*A) \oplus \oplus_\alpha \mathcal{L}(L^2(M_\alpha \times G_\alpha)) \tag{16}$$

extends to $\mathfrak{A}(M)$, the the norm closure of $\Psi_{\mathcal{V}}^0(M)$ acting on $L^2(M_0)$. Moreover, the structure theorems of [35][Theorem 4.4] (see also [32]) also show that the kernel of the map $\overline{\sigma}$ is given by the set of compact operators. (We are using here also the fact that solvable groups are amenable and hence that any irreducible $*$-representation of $\Psi_{\mathcal{V}}^0(M)$ is contained in one of the representations on $L^2(M_\alpha \times G_\alpha)$.) Therefore P is invertible modulo compact operators if, and only if, $\overline{\sigma}(P)$ is invertible. This completes the proof of Theorem 4.1. □

The first part of the above theorem has an elementary proof as follows. Choose $Q \in \overline{\Psi}_0$ such that $PQ - I$ and $QP - I$ are compact, using Corollary 4.9. Then $P_\alpha Q_\alpha - I = (PQ - I)_\alpha = 0$, by Theorem 4.8. Similary, $Q_\alpha P_\alpha - I = 0$. This proves that P_α is invertible, for all $\alpha \neq 0$. The ellipticity of P follows from classical results [28]. (See [42] for the details of this argument.) An elementary proof of the second part of the above theorem is usually obtained by constructing geometrically a bounded right s inverse of $\overline{\sigma}$ (s is defined on the range of $\overline{\sigma}$).

Earlier related results were obtained by [17, 30, 31, 34, 40, 41, 45, 47, 58, 59, 62].

It is interesting to notice that each of the operators P_α is G_α–invariant and "of the same kind" as the operator P, for example, if P is the Laplace operator associated to a compatible metric g, then P_α will be the Laplace operator corresponding to the induced metric on $M_\alpha \times G_\alpha$. See [3] for more results in this direction. This leads to an inductive procedure to study an operator $P \in \Psi_{\mathcal{V}}^\infty(M)$, which will be used, for example, in Section 6.

5. Examples

Let us discuss some examples of how the above theory can be used to study concrete examples. Most of these examples go back to Melrose [45]. Since M_0 is always the interior of M and the group G_0 is reduced to only one element, we shall typically assume below that $\alpha \neq 0$.

Example 5.1. Let M be a manifold with smooth, connected boundary ∂M, $M_0 = M \smallsetminus \partial M$, as before. On M we consider the set $\mathcal{V} = \mathcal{V}_b$ of vector fields that are *tangent* to ∂M. We impose no condition on these vector fields in the interior, as required by Axiom (iv) of the definition of a Lie manifold, Definition 2.1. Let $y_2, \ldots, y_n$ be some local coordinates on ∂M and let x denote the distance to the boundary. At the boundary $\partial M = \{x = 0\}$, a local basis of $\mathcal{V}_b$ is given by $x\partial_x, \partial_{y_2}, \ldots, \partial_{y_n}$.

An example of a compatible metric on M_0 is $g_0 = \frac{(dx)^2}{x^2} + h$, with h smooth on M. The resulting algebra $\mathrm{Diff}_{\mathcal{V}_b}(M)$ of differential operators is the algebra of totally characteristic differential operators. The metric on M_0 is that of a manifold with cylindrical ends. The resulting pseudodifferential calculus is the subalgebra of properly supported pseudodifferential operators in Melrose's b-calculus $\Psi_b(M)$. The Lie algebroid $A \to M$ is Melrose's compressed tangent bundle ${}^bTM$. The groupoid integrating ${}^bTM$ is obtained from Melrose's stretched b-product by removing the faces not intersecting the diagonal.

In this example $\{\alpha \neq 0\} = \{\partial M\}$ consists of exactly one element and $M_\alpha = \partial M$, $G_\alpha = \mathbb{R}$. The Fredholmness criteria were obtained in increasing generality in [30, 38, 46].

This example is basic in that it helps us understand easier other, more complicated examples. In the following examples we will indicate only what is different from the first example.

Example 5.2. Take now $\mathcal{V}_0$ to be the space of vector fields on M that vanish on ∂M. At the boundary $\partial M = \{x = 0\}$ a local basis is given by $x\partial_x, x\partial_{y_2}, \ldots, x\partial_{y_n}$. The resulting geometry is that of an asymptotically hyperbolic manifold. The strata different from M_0 are $M_\alpha = \{\alpha\}$ are parametrized by $\alpha \in \partial M$. The group $G_\alpha = T_\alpha(\partial M) \rtimes \mathbb{R}$ is a solvable Lie group with $t \in \mathbb{R}$ acting by dilation by e^t on $T_\alpha(\partial M)$.

Recently these manifolds have been used in Mathematical physics in connection to the AdS–CFT correspondence [6, 36, 15, 25]. Earlier, slightly larger larger algebras of pseudodifferential operators quantizing $\mathcal{V}_0$ were constructed in [41, 62] and called the "edge-calculus."

We now discuss an example that generalizes the manifolds Euclidean at infinity.

Example 5.3. Let us take now $\mathcal{V}_{sc}$ to be the space of vector fields on M that vanish on ∂M and have the property that their normal component to the boundary vanishes of second order at the boundary. At the boundary $\partial M = \{x = 0\}$ a local basis is given by $x^2\partial_x, x\partial_{y_2}, \ldots, x\partial_{y_n}$. The resulting geometry is that of an asymptotically flat manifold. As in the previous example, $\{\alpha \neq 0\} = \partial M$, each $M_\alpha = G_\alpha = T_\alpha(\partial M) \times \mathbb{R}$ is an abelian Lie group, and each P_α is G_α invariant.

This example is the best understood so far. For example, earlier versions of the pseudodifferential calculus were introduced by Parenti (called the "SG-calculus")

[56] and Melrose [45] (called the "scattering-calculus"). See [7] for an application of asymptotically flat manifolds to Quantum Gravity.

Here is now an example similar to that of asymptotically hyperbolic manifolds considered above. This example is relevant for the analysis on locally symmetric spaces and for boundary value problems on polyhedral domains.

Example 5.4. Let $\pi : \partial M \to B$ be a fibration, and let $\mathcal{V}_\pi$ be the space of vector fields on M that are tangent to the fibers of this fibration. We choose a system of coordinates at the boundary $\partial M = \{x = 0\}$ such that the fibration becomes a product in that neighborhood. Then a local basis of $\mathcal{V}_\pi$ on the domain of our coordinate chart is given by $x\partial_x, x\partial_{y_2}, \ldots, x\partial_{y_k}, \partial_{y_{k+1}}, \ldots, \partial_{y_n}$.

In this example, the set of non-zero parameters is $\{\alpha \neq 0\} = B$, the strata is given by $M_\alpha = \pi^{-1}(\alpha), \alpha \in B$, and $G_\alpha = T_\alpha B \rtimes \mathbb{R}$ is a solvable Lie group with $\mathbb{R}$ acting again by dilations. Earlier, slightly larger larger algebras of pseudodifferential operators quantizing $\mathcal{V}_0$ were constructed in [41, 62] and called the "edge-calculus."

We now include an example of a Lie manifold that is not type I. It a variation of the previous example. It is not clear how to generalize Theorem 4.1 to this example, although Fredholmness conditions can be obtained as in [32].

Example 5.5. Let $F \subset T\partial M$ be a foliation of the boundary of M. We assume that not all leaves of F are closed in M, to avoid trivialities. We take then $\mathcal{V} = \mathcal{V}_F$ to be the space of vector fields on M that are tangent to the leaves of F. No earlier pseudodifferential calculi on these manifolds were considered before.

We conclude with an example that generalizes our first example to manifolds with corners.

Example 5.6. Let M be a compact manifold with corners. We define $\mathcal{V} = \mathcal{V}_b$ to be the space of vector fields on M that are tangent to all hyperfaces of M, [42, 43]. In this example, $\{\alpha \neq 0\}$ is the set of faces H of maximal dimension of M (*i.e.,* the hyperfaces of M) and $M_H = H$ for any hyperface H. Finally, $G_\alpha = \mathbb{R}$. See also [42]. A Riemannian manifold isometric to M_0 with a compatible metric is called *a manifold with multi-cylindrical ends.*

We now discuss the Lie manifold with boundary associated to a convex polytope. They are type I.

Example 5.7. Let $\mathbb{P}$ be a simplex in $\mathbb{R}^N$. Let $(\Sigma(\mathbb{P}), \kappa)$ be its desingularization, where $(\Sigma(\mathbb{P}), \mathcal{V})$ is a Lie manifold with boundary, as in [2]. Then $\partial\Sigma(\mathbb{P})$ and the double of $\Sigma(\mathbb{P})$ are type I Lie manifolds. (The "double" of $\Sigma(\mathbb{P})$ is obtained by gluing two copies of $\Sigma(\mathbb{P})$ along their true boundary, *i.e.,* along the closure of the set of boundary points that correspond to each other and are not at infinity.)

6. Spectra

In this section we give an application of the Fredholmness conditions to the determination of the spectrum of the Dirac and Laplace operators on the manifolds arising in Example 5.6. In this section, we shall assume that M is a manifold with $\partial M \neq \emptyset$.

Let us consider for a moment the framework of 5.1, which is a particular case of Example 5.6. Let $P = \Delta_{M_0} - \lambda$. Then

$$P_\alpha = \Delta_{\partial M \times \mathbb{R}} = \Delta_{\partial M} - \partial_t^2 - \lambda.$$

Let $\hat{P}(\tau) = \Delta_{\partial M} + \tau^2 - \lambda$, be the Fourier transform of P_α in the t variable. This is what Melrose calls the "indicial family" associated to P. Since the spectrum of $\Delta_{\partial M}$ is

$$\sigma(\Delta_{\partial M}) = \{0, \lambda_1, \lambda_2, \dots\} \subset [0, \infty),$$

we obtain that $\hat{P}(\tau)$ is invertible for any $\tau \in \mathbb{R}$ if, and only if, $\lambda < 0$. Hence $\Delta_{M_0} - \lambda$ is Fredholm, if, and only if, $\lambda < 0$. This shows that $\sigma_e(\Delta_{M_0}) = [0, \infty)$. But then

$$[0, \infty) \subset \sigma_e(\Delta_{M_0}) \subset \sigma(\Delta_{M_0}) \subset [0, \infty)$$

and hence $\sigma(\Delta_{M_0}) = [0, \infty)$.

This argument generalizes to higher rank spaces [35] to prove the following result that was formulated as a conjectured in [45].

Theorem 6.1 (Lauter-Nistor)**.** *Assume M is as in Example 5.6 and $\partial M \neq \emptyset$. Then*

$$\sigma(\Delta_{M_0}) = [0, \infty).$$

We now extend the reasoning of the proof of the above theorem in [35] to study the Dirac operator. Recall that in this section we assume that $\partial M \neq \emptyset$.

Theorem 6.2. *Let M is as in Theorem 6.1 and $W \to M$ be a Clifford bundle over A^*. We assume no face of M has dimension zero. Let D_F be the Dirac operator on F with coefficients in $W|_F$ for any face $F \subset M$. We assume that* $\ker(D_F) = 0$, *for any $F \neq M$. Then each D_F is invertible and*

$$\sigma_e(D_M) = (-\infty, -c] \cup [c, \infty).$$

where $c^{-1} = \max\{\|D_H^{-1}\|\} > 0$, *for H ranging through the set of hyperfaces of M. Moreover, D_M is invertible if, and only if,* $\ker(D_M) = 0$.

Proof. We shall prove this by induction. If M has no boundary, then $\sigma_e(D_M) = \emptyset$ and D_M is invertible if, and only if, there exist no L^2-harmonic spinors (*i.e.,* $\ker(D_M) \neq 0$). This situation is excluded by our theorem since $\partial M \neq \emptyset$; it is needed, however, for the inductive hypothesis. If M has boundary, then $\sigma_e(D_M)$ is the spectrum of $D_1 := D_{\partial M} + c(dt)\partial_t$ acting on $L^2(\partial M \times \mathbb{R}, W_F)$, where t denotes the $\mathbb{R}$-component and $c(\omega)$ is the operator of Clifford multiplication by ω. We have $D_1^2 = -\partial_t^2 + D_{\partial M}^2$. Therefore $\sigma(D_1^2) = [c^2, \infty)$, where $c^{-1} = \|D_{\partial M}^{-1}\|$, is defined since there are no L^2-harmonic spinors on ∂M.

Let $V : L^2(\partial M \times \mathbb{R}, W_F) \to L^2(\partial M \times \mathbb{R}, W_F)$ be given by $V(u)(t) = c(dt)u(-t)$. Then $VD_1V^{-1} = -D_1$, and hence $\sigma(D_1)$ is symmetric with respect to 0. Hence $\sigma(D_1) = (-\infty, -c] \cup [c, \infty)$. This proves that $\sigma_e(D_M) = (-\infty, -c] \cup [c, \infty)$. In particular, since $0 \notin \sigma_e(D_M)$, we have that $0 \in \sigma(D_M)$ if, and only if, 0 is an eigenvalue of D_M.

The inductive step, in general, follows as exactly as in the case of a manifold with boundary, but replacing ∂M with a hyperface H of M. □

A similar reasoning gives the following.

Theorem 6.3. *We keep the same assumptions as in Theorem 6.2, except that we assume that* $\ker(D_F) = 0$ *for at least one face $F \neq M$. Then*

$$\sigma_e(D_M) = \mathbb{R}.$$

Proof. Let D_1 be the restriction of D_M to the groupoid corresponding to the face F. Then $\sigma(D_1) = \mathbb{R}$, as in the proof of Theorem 6.2. But $\sigma(D_1) \subset \sigma_e(D_M)$, by Theorem 4.1 (or by [35, 43]). □

We obtain the following corollary.

Corollary 6.4. *We continue to assume that M has no faces of dimension zero and keep the same notation as in Theorem 6.2. Then D_M is Fredholm if, and only if, D_F has no L^2-harmonic spinors, for any face $F \subset M$, $F \neq M$. Similarly, D_M is invertible if, and only if, D_F has no L^2-harmonic spinors, for any face $F \subset M$, including $F = M$.*

Proof. The first part is an immediate consequence of Theorem 6.2. Since $0 \notin \sigma_e(D_M)$, we have that $0 \in \sigma(D_M)$ if, and only if, 0 is an eigenvalue of D_M. □

The following theorem takes care of the case when there are faces of dimension zero, and hence completes our discussion.

Theorem 6.5. *Let M is as in Theorem 6.1 and $W \to M$ be a Clifford bundle over A^*. Assume M has faces of dimension zero. Let D be the Dirac operator with coefficients in W. Then*

$$\sigma_e(D) = \mathbb{R}.$$

Proof. Use the same reasoning as in [35]. Let F be a face of M of dimension zero (that is, F consists of one point). The restriction D_F of D to the (subgroupoid corresponding to the) face F is the Dirac operator over $F \times \mathbb{R}^n$ with coefficients in the pull-back of $W|_F$ to $F \times \mathbb{R}^n$, where n is the dimension of M. Since the spectrum of the Dirac operator on $\mathbb{R}^n$ is $\mathbb{R}$ (this can be proved using the argument in the proof of Theorem 6.2), it follows from Theorem 4.1 (or from [35, 43]) that $\mathbb{R} \subset \sigma_e(D)$. □

The results above extend to Dirac operators coupled with bounded potentials.

Our results on the spectrum of the Dirac operator are similar and compatible with the results of [14], where the spectrum of the Dirac operator on a manifold of finite volume is determined also in terms of the properties of the boundary at infinity. The setting in Bär's paper [14] is different from ours (although conformally equivalent). See also [1, 12, 13, 51].

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